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ON THE MOTION OF A HEAVY HOMOGENEOUS ELLIPSOID ON A FIXED HORIZONTAL PLANE*

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Periodic motions without sliding of a heavy homogeneous ellipsoid of nearly spherical shape on a horizontal plane is investigated. Existence of periodic motions of the ellipsoid relative to its center of mass is established on the basis of known solutions of the problem of the homogeneous sphere of steady motions on a plane. Periodic motions are determined, their stability is investigated, reaction of the plane is calculated, traces of the ellipsoid-plane contact point on the plane and ellipsoid surface and the ellipsoid orientation in absolute space are determined. Motion of such ellipsoid on an absolutely smooth plane is analyzed, and shown to be perpetual (for all times) and close to regular precession about the moment of momentum vector of constant length and precessing at constant angular velocity about the vertical to which it is inclined at a constant angle.

The problem of motion of a solid body on a fixed horizontal plane was investigated up to now in detail in /1 – 10/. The existence and stability of steady motions of a heavy solid body were the subject of detailed analysis whose most general results appear in /11 – 14/.

1. Let us consider the motions of an ellipsoid in the fixed system of coordinates $OXYZ$ with origin at some point O of the (supporting) horizontal plane, and its OZ axis directed vertically upward. We denote the unit vector of the OZ axis by $\mathbf{n}$ which is the unit vector of the external normal to the ellipsoid surface at point Q of the ellipsoid contact with the plane.

Axes of the coordinate system $Gxyz$ with origin at the ellipsoid center of mass G are directed along its principal axes. In that coordinate system the ellipsoid surface is defined by the equation

$$f \equiv x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \quad \mathbf{n} = \operatorname{grad} f/|\operatorname{grad} f| \tag{1.1}$$

and the radius vector $\mathbf{GQ}$ has the components x, y, z.

Orientation of the ellipsoid relative to the fixed coordinate system is specified by Euler's angles ψ, θ, φ. The relative orientation of the systems of coordinates $OXYZ$ and $Gxyz$ are also determined by the matrix of directional cosines $\| a_{ik} \|$ $(i, k = 1, 2, 3)$ that are expressed in the conventional manner in terms of Euler's angles, and

$$a_{31} = -b^2c^2x/\Delta, \quad a_{32} = -c^2a^2y/\Delta, \quad a_{33} = -a^2b^2z/\Delta \tag{1.2}$$
$$\Delta = (b^4c^4x^2 + c^4a^4y^2 + a^4b^4z^2)^{1/2}$$

Denoting by $\mathbf{V}_G$ and $\boldsymbol{\omega}$ the vectors of the ellipsoid center of mass velocity and of the vector of its instantaneous angular velocity, respectively, for the condition of absence of sliding we have

$$\mathbf{V}_G + \boldsymbol{\omega} \times \mathbf{GQ} = 0 \tag{1.3}$$

The equations of motion of the ellipsoid relative to its center of mass are expressed in the form of Gibbs – Appel equations /15/

$$(A + y^2 + z^2)\, p^{\cdot} - xyq^{\cdot} - xzr^{\cdot} = (B - C)\, qr + \tag{1.4}$$
$$(\boldsymbol{\omega}, \mathbf{GQ})(x^{\cdot} - yr + zq) - (\mathbf{GQ}, \mathbf{GQ}^{\cdot})\, p + ga^2(c^2 - b^2)yz/\Delta$$
$$\{pqr,\ xyz,\ abc,\ ABC\}$$
$$A = (b^2 + c^2)/5, \quad B = (c^2 + a^2)/5, \quad C = (a^2 + b^2)/5$$

where p, q, r are projections of vector $\boldsymbol{\omega}$ on axes Gx, Gy, Gz, g is the free fall acceleration, and A B, C are the principal central moments of inertia of the ellipsoid whose mass is assumed equal unity. The two omitted equations are obtained by the simultaneous cyclic permutation of symbols appearing in braces. The system of Eqs.(1.4) is closed by Poisson's equations

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$$a_{31}^{\cdot} = a_{32}r - a_{33}q, \quad a_{32}^{\cdot} = a_{33}p - a_{31}r, \quad a_{33}^{\cdot} = a_{31}q - a_{32}p$$

that express the OZ axis directional constancy in the absolute space. Using (1.1) and (1.2) we can write Poisson's equation in the form of the following three equations in x, y, z:

$$x^{\cdot} = yr - zq + \frac{a^2 - c^2}{a^2c^2}(x^2 - a^2)zq + \frac{b^2 - a^2}{b^2a^2}(x^2 - a^2)yr + \frac{c^2 - b^2}{c^2b^2}xyzp \quad \{xyz, pqr, abc\} \tag{1.5}$$

which by virtue of identity (1.1) are interdependent. The system of Eqs.(1.4) and (1.5) admits the energy integral

$$^1/_2 V_G^2 + {}^1/_2 (Ap^2 + Bq^2 + Cr^2) + gZ_G = h = \text{const} \tag{1.6}$$

where Z_G is the distance of the ellipsoid center from the horizontal plane, defined by

$$Z_G = (a^2a_{31}^2 + b^2a_{32}^2 + c^2a_{33}^2)^{1/2} \tag{1.7}$$

2. Setting in (1.1), (1.4), and (1.5) $a = b = c = R$ we obtain the problem of the homogeneous sphere moving without sliding on a plane.

Generally, excepting the case of pure rotation about the vertical axis, the vector of instantaneous angular velocity ω of the sphere is constant in magnitude and direction, the sphere center of mass moves uniformly and rectilinearly in a direction normal to ω, the trace of the contact point on the plane is a straight line and on the sphere surface is a circle of constant radius ρ, lying in a plane normal to ω at constant distance d from the sphere center (Fig.1), the normal reaction of the (supporting) plane is equal to the sphere weight, and the friction force is zero.

Suppose the ellipsoid is a three-axial one but close to a sphere of radius R. As the small parameter we take $\varepsilon = \max\{|a - b|/R, |b - c|/R, |c - a|/R\}$. Let us investigate the ellipsoid periodic motions relative to its center of mass in the case of $\varepsilon \neq 0$. When $\varepsilon = 0$ these motions become the defined above steady motions of a sphere.

It can be shown that Eqs.(1.4) and (1.5) are not changed by the following three substitutions:

$$\begin{aligned} &t, p, q, r, x, y, z \to -t, -p, q, r, -x, y, z \\ &t, p, q, r, x, y, z \to -t, p, -q, r, x, -y, z \\ &t, p, q, r, x, y, z \to -t, p, q, -r, x, y, -z \end{aligned} \tag{2.1}$$

These properties of symmetry will be subsequently used in investigations of the problem of existence of periodic motions and for constructing these by the method developed by Cesari /16/ and Hale /17/.

For the subsequent analysis it is expedient to substitute in Eqs.(1.4) and (1.5) the new variables ζ, ρ, γ for variables x, y, z. For this we first effect the following two substitutions:

$$x = ax', \quad y = by', \quad z = cz' \tag{2.2}$$

$$\begin{Vmatrix} x' \\ y' \\ z' \end{Vmatrix} = \begin{Vmatrix} \sin\beta & \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ -\cos\beta & \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ 0 & -\sin\alpha & \cos\alpha \end{Vmatrix} \begin{Vmatrix} \xi \\ \eta \\ \zeta \end{Vmatrix} \tag{2.3}$$

$$\sin\alpha = \sqrt{p^2 + q^2}/\omega, \quad \cos\alpha = r/\omega, \quad \sin\beta = q/\sqrt{q^2 + p^2}$$

$$\cos\beta = p/\sqrt{q^2 + p^2} \quad \left(\omega = \sqrt{p^2 + q^2 + r^2}\right)$$

The substitution (2.3) of variables effects the passing from the system of coordinates $Gx'y'z'$ to the system $G\xi\eta\zeta$ whose axis $G\zeta$ is parallel to vector ω, the axis $G\xi$ is normal to the plane $Gz'\zeta$, and axis $G\eta$ forms an obtuse angle with axis Gz'; α is the angle between vector ω and axis Gz', and β is the angle between the projection of ω on plane $Gx'y'$ and axis Gx' (see Fig.2, where vector ω is shown passing through the center of mass , while in fact it passes throught the point Q of contact between the ellipsoid and the plane).

A further substitution of variables using formulas

$$\xi = \rho\sin\gamma, \quad \eta = \rho\cos\gamma \tag{2.4}$$

yields Eq.(1.1) and equations for ζ and γ of the form

$$\zeta^2 + \rho^2 = 1 \tag{2.5}$$

$$\begin{aligned} \zeta^{\cdot} = {} & [(c - b)rqx' + (a - c)pry' + (b - a)qpz']h_1 + \\ & [(c - b)py'z' + (a - c)qz'x' + (b - a)rx'y']h_2 + \ldots \end{aligned} \tag{2.6}$$

$$\gamma^{\cdot} = \omega - (F_1\cos\gamma + F_2\sin\gamma)/(\omega\rho) + \ldots \tag{2.7}$$

$$h_1 = (14 - 15\zeta^2)/(7\omega R), \quad h_2 = 2\zeta(2 - 5\zeta^2 - 5g/(\omega^2R))/(7R) \tag{2.8}$$

$$F_1 = p^{\cdot}(\mathrm{ctg}\,\alpha\eta + \zeta)\sin\beta - q^{\cdot}(\mathrm{ctg}\,\alpha\eta + \zeta)\cos\beta - \omega\sin\beta g_1 + \omega\cos\beta g_2$$

$$F_2 = p^{\cdot}(\mathrm{ctg}\alpha\sin\beta\xi - \cos\alpha\cos\beta\zeta) - q^{\cdot}(\mathrm{ctg}\alpha\cos\beta\xi + \cos\alpha\sin\beta\zeta) + r^{\cdot}\sin\alpha\zeta + \omega\cos\alpha\cos\beta g_1 + \omega\cos\alpha\sin\beta g_2 - \omega\sin\alpha g_3$$

$$g_1 = 2(c-b)x'y'z'p/R + (a-c)(2x'^2-1)z'q/R + (b-a)\times(2x'^2-1)y'r/R\{g_1g_2g_3, x'y'z', pqr\}$$

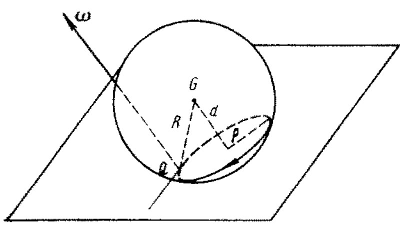

Fig.1

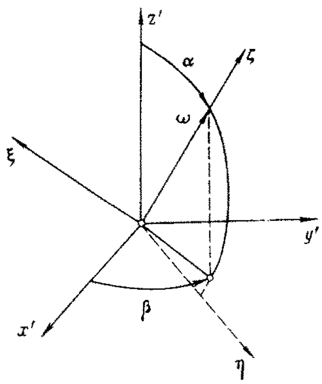

Fig.2

The quantities $x', y', z', \xi, \eta, \alpha, \beta$ must be expressed in terms of $p, q, r, \zeta, \rho, \gamma$ in conformity with formulas (2.3) and (2.4). The dots (in formulas (2.6) and (2.7) denote terms of order ε^2 and higher.

To define the ellipsoid motion in the new variables it is necessary in the system of Eqs.(1.4) and (1.5) to substitute Eqs.(2.6) and (2.7) for Eqs.(1.5), and the equality (2.5) for (1.1).

When $\varepsilon = 0$ the quantities $p, q, r, \zeta, \rho, \gamma^{\cdot}$ in all solutions of the transformed system of equations are constant, and the mechanical and geometric meaning of the new variables ζ, ρ, γ becomes clear. It we take the sphere radius as the unit of length, then ρ is the radius of the circle representing the trace of the contact point on the sphere and $|\zeta|$ is the distance d of the plane of that circle from the sphere center (Fig.1); and γ is the angle between the projection of GQ on the plane of the trace circle and some fixed in that plane direction, with $\gamma^{\cdot}$ equal to the angular velocity ω of motion of the contact point along its trace on the sphere.

3. On the basis of formulas (2.2) – (2.4) it is possible to obtain from the energy integral (1.6) and the geometric integral (2.5) ζ and ρ in the form of functions of variables p, q, r, γ and parameter h. Then, substituting into Eqs.(1.4) x, y, z expressed in terms of ζ, ρ, γ and solving them for $p^{\cdot}, q^{\cdot}, r^{\cdot}$, we obtain a system of three equations whose right-hand sides are functions of p, q, r, γ and parameter h and are 2π-periodic with respect to the angular variable γ

To obtain the geometric pattern of motion we pass again, using (2.7), to the new independent variable γ. Calculations show that then the system of equations for p, q, r is of the form

$$\frac{dp}{d\gamma} = \frac{5}{7\omega R}\Big\{(c-b)\Big(qrx'^2 + 2p^2z'y' + 2p\omega\zeta x'y'z' + \frac{2g}{R}y'z' + \frac{2}{5}qr\Big) + (a-c)(rpx'y' + 2qpx'z' + 2q\omega\zeta(x'^2-1)z') + (b-a)(pqx'z' + 2rpx'y' + 2r\omega\zeta(x'^2-1)y')\Big\} + \ldots \tag{3.1}$$
$$\{pqr, x'y', z', abc\}$$

For the determination of the 2π-periodic in γ solutions of system (3.1) we make use of the property of symmetry of (2.1). We shall consider only the second symmetry property, since the other reduce to it by a cyclic re-designation of axes of system $Gxyz$.

Equations (3.1) have the property E (see /17/) with respect to $\mathbf{Q} = \mathrm{diag}(1, -1, 1)$ which means that it is not affected by the substitution $\gamma, p, q, r \to -\gamma, p, -q, r$. Periodic solutions of system (3.1) are obtained in conformity with the following algorithm /17/.

Consider the system of equations

$$\mathbf{z}^{\cdot} = \varepsilon\mathbf{Z}(t, \mathbf{z}, \varepsilon) \tag{3.2}$$

where ε is a small parameter, and function $\mathbf{Z}$ is analytic in $\mathbf{z}$ and ε, and is 2π-periodic in t

and continuous. When $\varepsilon = 0$ solutions of system (3.2) are constant. All of its periodic solutions generated when $\varepsilon \neq 0$ from its constant solutions can be obtained as follows. We set

$$\mathbf{z}^{(0)} = \mathbf{a}^*, \quad \mathbf{z}^{(k+1)} = a^* + \varepsilon \int [\mathbf{Z}(t, \mathbf{z}^{(k)}(t), \varepsilon) - \langle \mathbf{Z}(t, \mathbf{z}^{(k)}(t), \varepsilon) \rangle] \, dt \tag{3.3}$$

where the angle brackets denote averaging with respect to the variable t. Successive approximations of (3.3) determine the approximate value of the periodic function $\mathbf{z}(t, \mathbf{a}^*, \varepsilon)$ with an accuracy to terms of order ε^{k+1} inclusive. This value is a function of the arbitrary constant vector $\mathbf{a}^*$.

Let us construct the system of branching equations

$$\langle \mathbf{Z}(t, \mathbf{z}^{(k)}(t, \mathbf{a}^*, \varepsilon), \varepsilon) \rangle = 0 \tag{3.4}$$

which is used for determining the generating solution, i.e. the constant vector $\mathbf{a}^*$ as a function of ε. If system (3.4) has a solution for which the Jacobian is nonzero, then by substituting that solution into $\mathbf{z}^{(k+1)}$ we obtain a 2π-periodic solution of system (3.2) accurate to ε^{k+1} inclusive.

When $k = 0$ we have

$$\mathbf{z}^{(1)} = \mathbf{a}^* + \varepsilon \int [\mathbf{Z}(t, \mathbf{a}^*, 0) - \langle \mathbf{Z}(t, a^*, 0) \rangle] \, dt \tag{3.5}$$

where $\mathbf{a}^*$ satisfies the system of equations

$$\langle \mathbf{Z}(t, \mathbf{a}^*, 0) \rangle = 0 \tag{3.6}$$

Frequently (and that is the most interesting case in applications) the Jacobian for solutions of system (3.4) vanishes. Then the property E of system (3.2) enables us to obtain the sufficient conditions of periodic solutions existence that correspond to some particular choice of vector $\mathbf{a}^*$. Thus, if the j-th element of the diagonal matrix Q is equal $+1$, the j-th equation of the branching system (3.4) is satisfied (in all approximations with respect to ε) by any vector $\mathbf{a}^*$ for which $Q\mathbf{a}^* = \mathbf{a}^*$ /17/.Because of this a number of equations vanish from (3.4), and this enables us to determine the conditions of existence of periodic solutions when the Jacobian for some solutions of system (3.4) is zero.

Calculations show that the indicated Jacobian of the branching system (3.6) is identically zero in the case of Eqs.(3.1). However, since system (3.1) has the property E with respect to matrix Q, we can set

$$\mathbf{a}^{*T} = (p^*, 0, r^*) \tag{3.7}$$

Since for such selection of $\mathbf{a}^*$ the first and third of equations of the branching system (3.4) is identically satisfied, it is only necessary to analyze the second equation for arbitrary p^*, r^*.

Restricting the solution derivation to the first approximation with respect to γ, we consider the second of Eqs.(3.6) which is obtained by averaging the right-hand side of the second equation of system (3.1) with respect to γ with $p = p^*$, $q = 0$, $r = r^*$. In the derivation of the first approximation of p, q, r the quantities ζ and ρ appearing in x', y', z' can be assumed constant, i.e. ζ^* and ρ^*. The latter satisfy relations $\zeta^{*2} + \rho^{*2} = 1$, $\rho^* > 0$, being otherwise arbitrary. The averaging yields the branching equation

$$p^* r^* F(\rho^*, \omega^*) = 0 \tag{3.8}$$

$$F(\rho^*, \omega^*) = 10\rho^{*4} - 5(6\sigma^* + 5)\rho^{*2} + 4(5\sigma^* + 6)$$

$$\sigma^* = \frac{g}{\omega^{*2} R}, \quad \omega^* = \sqrt{p^{*2} + r^{*2}}$$

To analyze Eq.(3.8) we first consider the equation $F(\rho^*, \omega^*) = 0$ which implies that

$$\omega^{*2} = \frac{10(3\rho^{*2} - 2) g}{(10\rho^{*4} - 25\rho^{*2} + 24) R} \tag{3.9}$$

where ρ^* varies within the interval from zero to unity. Function $F(\rho^*, \omega^*)$ can only vanish when the inequalities $\rho^* > \sqrt{2/3}$ and $\omega^* < \frac{1}{3}\sqrt{10g/R}$ are satisfied, and the derivative $\partial F / \partial \omega^* \neq 0$ when $F(\rho^*, \omega^*) = 0$. Hence the roots of equation $F(\rho^*, \omega^*) = 0$, if any, are simple.

The branching equation (3.8) has solutions of three types for which the Jacobian is nonzero. For solutions of the first type $r^* = 0$ and p^* is arbitrary but such that $F(\rho^*, \omega^*) \neq 0$; for solutions of the second type $p^* = 0$ and r^* is arbitrary but $F(\rho^*, \omega^*) \neq 0$; for solutions of the third type $F(\rho^*, \omega^*) = 0$ and $p^* \neq 0$, $r^* \neq 0$. When $\varepsilon \neq 0$, these solutions generate 2π-periodic in γ solutions of the system of Eqs.(3.1).

The proof of existence of periodic solutions of Eqs.(3.1) can be similarly obtained by using the property E relative to matrices $\operatorname{diag}(-1,1,1)$ and $\operatorname{diag}(1,1,-1)$. In the latter case it is necessary first, to substitute the quantity $\pi/2-\gamma$ for γ.

There exist, thus, two different sets of periodic motions of the ellipsoid. Periodic motions of the first set pass for $\varepsilon=0$ to motions in which vector ω is parallel to one of the ellipsoid axes. In such motions when $\varepsilon=0$ the rolling and rotating ellipsoid (sphere) contacts the horizontal plane along the (periphery of) cross section parallel to its principal cross section. Periodic motions of the first set exist if $\omega^*>{}^1/_3\sqrt{10g/R}$, and ρ^* assumes any value from zero to unity, or when $\omega^*<{}^1/_3\sqrt{10g/R}$, and ρ^* and ω^* are not related by formula (3.9). In the case of periodic motions of the second set with $\varepsilon=0$ the projection of ω on one of the ellipsoid axes is zero and on the other two nonzero. In such periodic motions when $\varepsilon=0$ the ellipsoid (sphere) contacts the plane along the (periphery of) of its cross section parallel to one of its axes and intersecting the other two axes. Periodic motions of the second set exist if $\omega^*<{}^1/_3\sqrt{10g/R}$ and $\rho^*>\sqrt{{}^2/_3}$, provided that the relations of the equality type, i.e. (3.9), which link the generating values of parameters ω and ρ. Periodic motions of the second set will not be further considered, since they can be realized only in exceptional cases.

4. Let us construct the periodic solutions of the first set that for $\varepsilon=0$ becomes the generating solution

$$p=p^*,\ q=0,\ r=0,\ \rho=\rho^*,\ \zeta=\zeta^* \tag{4.1}$$

For definiteness we assume p^* to be positive. To obtain the first approximation of solution it is necessary to substitute in the right-hand sides of system (3.1) the values of p, q, r, x', y', z' that correspond to the generating solution (4.1), reject terms of higher order, above first, with respect to ε; and integrate. As the result we have

$$p=p^*-\frac{5\omega^*\rho^{*2}}{14R}(c-b)(2-\rho^{*2}+\sigma^*)\cos 2\gamma \tag{4.2}$$

$$q=-\frac{5\omega^*\rho^*\zeta^*}{42R}\{12(a-c)\sigma^*\sin\gamma+(c-b)[3(\rho^{*2}-4)\sin\gamma-\rho^{*2}\sin 3\gamma]\}$$

$$r=\frac{5\omega^*\rho^*\zeta^*}{42R}\{12(b-a)\sigma^*\cos\gamma+(c-b)[3(\rho^{*2}-4)\cos\gamma+\rho^{*2}\cos 3\gamma]\}$$

If the case of an ellipsoid of revolution $(b=c)$ the projection p of the instantaneous angular velocity vector ω on the axis of symmetry is in the first approximation constant, the end point of vector ω lies in a plane normal to the axis of symmetry and moves about the latter in a circle or radius $10\,|\,a-c\,|\,\omega^*\rho^*\zeta^*\sigma^*/(7R)$ at the angular velocity $\dot\gamma$.

Let us determine the coordinates x, y, z of the contact point Q of the ellipsoid and the plane in terms of γ. First, using (2.7) we determine function ζ in the first approximation with respect to ε

$$\zeta=\zeta^*-\frac{c-b}{14R}\zeta^*\rho^{*2}(5\rho^{*2}-5\sigma^*-3)\cos 2\gamma \tag{4.3}$$

Function ρ is then determined using (2.5).

It follows from (4.3) and (2.5) that in the case of an ellipsoid of revolution $(c=b)$ the quantities ζ and ρ are constant in the first approximation or, when the ellipsoid rolling is such that it touches the plane along (the periphery of) its principal cross section Gyz ($\zeta^*=0$), or, when ρ^* and ω^* are linked by the relationship $5\rho^{*2}-5\sigma^*-3=0$ which is only possible when the inequalities $\omega^*>\sqrt{5g/(2R)}$ and $\rho^*>\sqrt{{}^3/_5}$ are simultaneously satisfied.

Now, from (2.3), (2.4), (2.5), (4.2), and (4.3) we obtain x', y', z' with an accuracy within terms of first order with respect to ε

$$x'=\zeta^*+\frac{\zeta^*\rho^{*2}}{42R}[30(b+c-2a)\sigma^*+(c-b)(5\rho^{*2}-15\sigma^*-51)\cos 2\gamma] \tag{4.4}$$

$$y'=-\rho^*\sin\gamma-\frac{\zeta^{*2}\rho^*}{84R}\{120(a-c)\sigma^*\sin\gamma+(c-b)[3(5\rho^{*2}+5\sigma^*-37)\sin\gamma+(5\rho^{*2}-15\sigma^*-9)\sin 3\gamma]\}$$

$$z'=-\rho^*\cos\gamma+\frac{\zeta^{*2}\rho^*}{84R}\{120(b-a)\sigma^*\cos\gamma+(c-b)[3(5\rho^{*2}+5\sigma^*-37)\cos\gamma-(5\rho^{*2}-15\sigma^*-9)\cos 3\gamma]\}$$

To obtain x, y, z it remains to use formulas (2.2).

The form of point Q trace on the ellipsoid surface is shown in Fig.3. It is contained between two planes parallel to plane Gyz separated by the distance

$$\Delta x=\rho^{*2}\,|\,(c-b)\,\zeta^*\,(5\rho^{*2}-15\,\sigma^*-51)\,|/21$$

The trace touches these planes when $\sin 2\gamma=0$. The direction of motion of the contact point Q along its trace on the ellipsoid is indicated in Fig.3 by an arrow.

5. The time period T of the constructed periodic motions of the ellipsoid is obtained from (2.7) in the form

$$T=\frac{2\pi}{\omega^*}\left[1-\frac{5\zeta^{*2}\sigma^*}{7R}(b+c-2a)+\ldots\right]$$

From (2.7) we also obtain angle γ as a function of time

$$\gamma=\gamma^*+\frac{2\pi}{T}t+\frac{c-b}{28R}[5\rho^{*4}-15(\sigma^*+2)\rho^{*2}+2(5\sigma^*+3)]\cos\frac{4\pi t}{T}+\ldots$$

where γ^* is an arbitrary constant.

6. Let us determine the trace of the contact point Q on the plane. Let X, Y be the coordinates of point Q on the plane, and δ be the angle between the tangent to its trace on the plane and the OX axis of the fixed coordinate system (or, what is the same by virtue of absence of sliding, δ is the angle between the tangent to the trace on the ellipsoid at point Q and the OX axis). Then

$$dX/ds=\cos\delta,\quad dY/ds=\sin\delta \tag{6.1}$$

where ds is an element of length of the trace arc on the plane (or on the ellipsoid).

Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors of velocity and acceleration of point Q in its translation along the trace on the ellipsoid, $\mathbf{QP}$ be the vector drawn from point Q to the center of the trace curvature, and k be the trace curvature at point Q. Then

$$v=\sqrt{x^{\cdot 2}+y^{\cdot 2}+z^{\cdot 2}},\quad w=\sqrt{x^{\cdot\cdot 2}+y^{\cdot\cdot 2}+z^{\cdot\cdot 2}} \tag{6.2}$$
$$\mathbf{QP}=\frac{\mathbf{v}^2}{(\mathbf{v}\times\mathbf{w})^2}(\mathbf{v}\times\mathbf{w})\times\mathbf{v},\quad k=\frac{w}{v^2}\sin(\mathbf{v},\mathbf{w})$$

Note that for the determination of angle δ /3/

$$\delta^{\cdot}=\omega_B-k_g v;\quad \omega_B=-(\boldsymbol{\omega},\mathbf{n}),\quad k_g=k\sin(\mathbf{QP},\mathbf{n}) \tag{6.3}$$

where ω_B is the angular velocity of the ellipsoid rotation and k_g is the geodesic curvature of the trace at point Q /18/.

If we pass to the new independent variable γ, the differential equation for δ for the derived periodic motions may be written as

$$\frac{d\delta}{d\gamma}=\mu_1\rho^*R+2\mu_2\cos 2\gamma+\ldots,\quad \mu_1=5\frac{b+c-2a}{7\rho^*R^2}\zeta^*\sigma^* \tag{6.4}$$
$$\mu_2=\frac{c-b}{14R}\zeta^*(5\sigma^*+24)$$

Noting that $ds/d\gamma$ differs from ρ^*R by a quantity of the first order of smallness and setting the initial value of δ at zero, we obtain from (6.4)

$$\delta=\mu_1 s+\mu_2\sin(2s/(\rho^*R))+\ldots \tag{6.5}$$

Retaining in the right-hand sides of Eqs.(6.1) terms of order of smallness not higher than in the first, we obtain equations whose solutions accurate to within terms of the first order of smallness are of the form

$$X-X^*=\mu_1^{-1}\sin\mu_1 s+{}^1/_2\,\mu_2\,\rho^*R\sin\mu_1 s\cos(2s/(\rho^*R)) \tag{6.6}$$
$$Y-Y^*=-\mu_1^{-1}\cos\mu_1 s-{}^1/_2\,\mu_2\rho^*R\cos\mu_1 s\cos(2s/(\rho^*R))$$

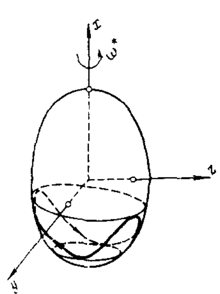

Fig.3

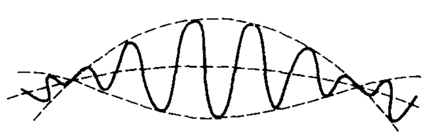

Fig.4

where X^*, Y^* are arbitrary constants. Equalities (6.6) define in parametric form the equation for the trace of contact point Q on the plane.

Let us consider the simplest particular cases.

If $\mu_2=0$ (an ellipsoid of revolution $c=b$), the trace is an arc of a circle of radius $|\mu_1|^{-1}$ with the center at point X^*, Y^* (when $\zeta^*=0$ the ellipsoid rolls touching the plane by /the periphery of/

its principal cross section and the trace of the contact point is a straight line). As expected, in the case of a sphere ($b = c = a$) the circle degenerates into a straight line. The direction of motion of point Q is determined by the sign of $\zeta^* (b - a)$. For instance, when $\zeta^* < 0$ (as in Fig.3), then for $a > b$ (a prolate ellipsoid) the contact point motion along its trace circle is counterclockwise, and in the case of $a < b$ (an oblate ellipsoid) it is clockwise.

Let $\mu_1 = 0$. This is possible when the ellipsoid semiaxes satisfy the relation $b + c = 2a$ or when $\sigma^* = 0$, i.e. when the ellipsoid is weightless. The latter case is not a mathematical abstraction, since, for example, to it reduces the case in which σ^* is a quantity of an order of smallness not lower than the first, which shows that the ellipsoid potential energy is considerably lower than its kinetic energy of rotation about its center of mass. When $\mu_1 = 0$ we have

$$Y = Y^* - {}^1/_2\mu_2\rho^* R \cos [2 (X - X^*)/(\rho^* R)]$$

i.e. the contact point trace on the plane is a sine curve. This result was obtained in /3/ for a weightless ellipsoid of revolution close to a sphere.

Generally, when $\mu_1 \neq 0, \mu_2 \neq 0$ the trace on the plane is an arc of "spoiled" circle of radius $|\mu_1|^{-1}$. Small oscillations of slowly varying amplitude ${}^1/_2\, \mu_2\rho^* R \sin \mu_1 s$ are superposed on the circle. The contact point trace on the plane is shown in Fig.4 in a distorted scale.

7. Let us determine the ellipsoid orientation relative to the fixed coordinate system $OXYZ$ for the derived periodic motions. The nutation angles θ and of proper rotation φ are calculated using formulas (1.2), (2.2) and (4.4), and the expressions for directional cosines in terms of Euler's angles

$$a_{31} = \sin\theta \sin\varphi,\ a_{32} = \sin\theta\cos\varphi,\ a_{33} = \cos\theta \tag{7.1}$$

Carrying out the necessary calculations, we obtain

$$a_{31} = -\zeta^* - \frac{b+c-2a}{14R}\zeta^*\rho^{*2}(10\sigma^* + 7) - 5\,\frac{c-b}{42R}\zeta^*\rho^{*2}(\rho^{*2} - 3\sigma^* - 6)\cos 2\gamma + \ldots \tag{7.2}$$

$$\begin{aligned} a_{32} = {} & \rho^*\sin\gamma + \frac{\rho^*}{28R}\{-5(c-b)\rho^{*4} - [40(a-c)\sigma^* + \\ & (c-b)(5\sigma^* - 49) - 28(b-a)]\rho^{*2} + 40(a-c)\sigma^* - 28(b-a) + \\ & (c-b)(5\sigma^* - 37)\}\sin\gamma - \frac{c-b}{84R}\rho^*[5\rho^{*4} - 5(3\sigma^* + 7)\rho^{*2} + \\ & 3(5\sigma^* + 3)]\sin 3\gamma + \ldots \end{aligned} \tag{7.3}$$

$$\begin{aligned} a_{33} = {} & \rho^*\cos\gamma + \frac{\rho^*}{28R}\{5(c-b)\rho^{*4} + [40(b-a)\sigma^* + \\ & (c-b)(5\sigma^* - 49) - 28(a-c)]\rho^{*2} - 40(b-a)\sigma^* + \\ & 28(a-c) - (c-b)(5\sigma^* - 37)\}\cos\gamma - \\ & \frac{c-b}{84R}\rho^*[5\rho^{*4} - 5(3\sigma^* + 7)\rho^{*2} + 3(5\sigma^* + 3)]\cos 3\gamma + \ldots \end{aligned} \tag{7.4}$$

Formulas (7.2)—(7.4) define the dependence of the orientation of the fixed vertical line relative to the ellipsoid on γ. It follows from (7.2) that the angle between the vertical and the Gx axis varies about its constant value with a frequency close to twice the angular velocity of the unperturbed motion (when $\varepsilon = 0$). The oscillation amplitude of that angle cosine is $5\rho^{*2}|(c-b)\zeta^*(\rho^{*2} - 3\sigma^* - 6)|/42R$. In the case of an ellipsoid of revolution ($c = b$) these oscillations are absent. They are also, obviously, absent when the rolling ellipsoid touches the plane along (the periphery of) its principal cross section ($\zeta^* = 0$).

The precession angle ψ can be obtained by integrating Euler's kinematic equations. However it is possible, in conformity with /3/, to avoid integration. Let $\mathbf{e}$ and $\boldsymbol{\tau}$ be unit vectors directed along the line of nodes and that of the tangent to the trace of point Q on the ellipsoid, respectivley. In the coordinate system $Gxyz$ we have

$$\mathbf{e}^T = (\cos\varphi, -\sin\varphi, 0),\ \boldsymbol{\tau}^T = (x^\cdot/v,\ y^\cdot/v,\ 0) \tag{7.5}$$

The angle between $\mathbf{e}$ and $\boldsymbol{\tau}$ is equal $\psi - \delta$. Hence it follows from (7.5) that

$$\cos(\psi - \delta) = (x^\cdot\cos\varphi - y^\cdot\sin\varphi)/v \tag{7.6}$$

In determining angle ψ with an error of order ε it is necessary to set $\delta = \mu_1\rho^* R\gamma$ and calculate the right-hand side of (7.6) with $\varepsilon = 0$. We obtain

$$\cos(\psi - \mu_1\rho^* R\gamma) = -\zeta^*\cos\gamma / \sqrt{1 - \rho^{*2}\cos^2\gamma} \tag{7.7}$$

Angle ψ is determined from this with an error of order ε in the interval of γ variation of order ε^{-1}. Angles θ and φ are determined from (7.2)—(7.4) for any γ with an error of order ε^2.

8. Consider the normal reaction N and friction force F at the ellipsoid point of contact with the plane. The determination of these quantities is the necessary part of solution of the motion problem of a solid body on a fixed or moving surface. The condition that the moving body and the surface have at every instant of time one common point implies a nonholding link, so that at the instant when the normal reaction of the surface passes through zero, the body may free itself of that link. In the investigated here problem the ellipsoid may jump above the plane, and the motions determined on the assumption of contact between the ellipsoid and plane loose their meaning.

Further, the friction force must be determined, since its value obtained on the assumption of absence of slip may prove to exceed the force developed at the contact point of the body and plane for a given specific friction coefficient. This means that slip will occur, and the calculations based on the assumption of its absence are devoid of real mechanical meaning.

We obtain the plane reaction using the theorem on the change of momentum

$$\mathbf{W}_G = -g\mathbf{n} + \mathbf{N} + \mathbf{F} \tag{8.1}$$

where $\mathbf{W}_G$ is the acceleration of the ellipsoid center of mass. Using (1.3), (4.2), (7.2)—(7.4), from (8.1) we obtain

$$N = g + {}^1/_7 (c-b)\, \omega^{*2} \rho^{*2} [5(\sigma^* - 1)\rho^{*2} - (5\sigma^* + 9)] \times \cos 2\gamma + \ldots \tag{8.2}$$

It is then possible to calculate the friction force which is a quantity of the first order of smallness with respect to ε.

Formula (8.2) implies that for a given ω^* and fairly small $|c-b|$ the normal reaction N differs only slightly from the ellipsoid weight, hence jumping of the ellipsoid over the plane does not occur. The friction force necessary for preventing slipping of the ellipsoid can be obtained even at low friction coefficients.

For given semiaxes of the ellipsoid its angular velocity ω^* must not be high, since in conformity with (8.2) the normal reaction N may be zero for considerable ω^* (of order $\sqrt{g/\varepsilon R}$).

For the obtained periodic motions of the ellipsoid the velocity of its center of mass is defined by

$$V_G = \sqrt{(b^2+c^2)/2}\, \rho^*\omega^* \left[1 - \frac{c-b}{14R}(2\rho^{*2} + 5\sigma^* - 4)\cos 2\gamma + \ldots\right] \tag{8.3}$$

Projection p of angular velocity $\boldsymbol{\omega}$ on the Gx axis, the angle between that axis and the vertical, the center of mass velocity, and the normal reaction N of the plane reach their extremal values simultaneously at $\sin 2\gamma = 0$). At these instants of time the trajectory of contact point Q pass on the ellipsoid and the plane through extremal values.

9. Let us investigate the stability of the obtained periodic motions of the ellipsoid with respect to the perturbed variables p, q, r, ζ. Presence of stability means that at small initial deviations of quantities p, q, r, ζ from their values in unperturbed motion defined by formulas (4.2) and (4.3), the trace of the contact point on the ellipsoid surface, the instantaneous angular velocity vector and its orientation relative to the ellipsoid (hence relative to the absolute space) vary only slightly.

We investigate stability in the first approximation. The quantity γ can be taken, by virtue of its monotonic increase with increasing t, as the independent variable. Denoting by x_1, x_2, x_3, x_4 the perturbations of quantities p, q, r, ζ, respectively, and linearizing Eqs. (2.7) and (3.1), we obtain

$$d\mathbf{x}/d\gamma = \mathbf{A}_1(\gamma)\,\mathbf{x} + \ldots \quad (\mathbf{x}^T = (x_1, x_2, x_3, x_4)) \tag{9.1}$$

where the elements of matrix $\mathbf{A}_1$ are of the first order of smallness with respect to ε, are 2π-periodic in γ, and contain ρ^*, ω^* as parameters.

We present the fundamental matrix $\mathbf{X}(\gamma)$ normalized by the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the unit matrix, in the form of series

$$\mathbf{X}(\gamma) = \mathbf{E} + \mathbf{X}_1(\gamma) + \ldots \tag{9.2}$$

where $\mathbf{X}_i(0) = 0$. From (9.1) and (9.2) we have

$$\mathbf{X}_1(\gamma) = \int_0^\gamma \mathbf{A}_1(t)\,dt, \quad \mathbf{X}(2\pi) = \mathbf{E} + 2\pi\langle\mathbf{A}_1\rangle + \ldots$$

When $\varepsilon = 0$, the characteristic indices are obviously zero. Hence the characteristic indices of system (9.1) can be determined as the eigenvalues of matrix

$$(2\pi)^{-1}\ln \mathbf{X}(2\pi) = (2\pi)^{-1}\ln(\mathbf{E} + 2\pi\langle\mathbf{A}_1\rangle + \ldots) = \langle\mathbf{A}_1\rangle + \ldots$$

The problem of stability in the first approximation is, thus, reduced to the investigation of stability of system (9.1) with matrix $A_1(\gamma)$ averaged with respect to γ. Calculations show that the averaged system (9.1) is

$$dx_1/d\gamma = 0, \quad dx_2/d\gamma = (a-c)\, F(\rho^*, \omega^*)\, x_3/(14R)$$
$$dx_3/d\gamma = (b-a)\, F(\rho^*, \omega^*)\, x_2/(14R), \quad dx_4/d\gamma = 0 \tag{9.3}$$

where terms of order higher than the first with respect to ε have been omitted. Function $F(\rho^*, \omega^*)$ do not vanishes in conformity with the condition of existence of the investigated set of periodic motions. Equation (9.3) is such that

$$\lambda^2\left[\lambda^2 - \frac{(a-c)(b-a)}{196R^2} F^2(\rho^*, \omega^*)\right] = 0 \tag{9.4}$$

When $(a-c)(b-a) > 0$, Eq.(9.4) has a root with a positive real part, while for $(a-c)(b-a) < 0$ it has a pair of pure imaginary roots and a pair of zero roots to which obviously correspond simple elementary divisors.

Hence in the case of fairly small distinction of the ellipsoid from a sphere the following statement is true. If in the generating motion the ellipsoid rotates about the axis parallel to its semimean axis, the considered periodic motion is unstable, if however the rotation is about the semimajor or semiminor axes of the ellipsoid, there is stability in the first approximation (and when terms of second and higher order with respect to ε are neglected in (9.1)).

10. Let the plane on which the ellipsoid is moving be absolutely smooth, then the considered mechanical system is holonomic and has five degrees of freedom. For the investigation of motion we can take as the independent generalized coordinates three Euler's angles and the two coordinates X_G, Y_G of the ellipsoid center of mass relative to the fixed coordinate system $OXYZ$. The coordinate Z_G is expressed in terms of Euler's angles in conformity with formulas (1.7) and (7.1). The first three of $X_G, Y_G, \psi, \theta, \varphi$ are cyclic. This implies that $X_G^{\cdot}, Y^{\cdot}$ are constant, i.e. the projection of the ellipsoid center on the horizontal plane moves uniformly and rectilinearly; moreover, the momentum I_3 that corresponds to the generalized coordinate ψ is also constant, which shows the invariability of projection on the vertical of the ellipsoid momentum vector relative to its center of mass, i.e. $I_3 = Apa_{31} + Bqa_{32} + Cra_{33} = \text{const}$.

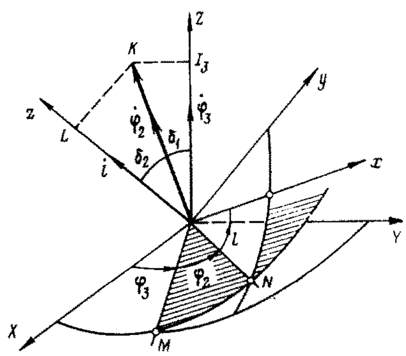

Fig.5

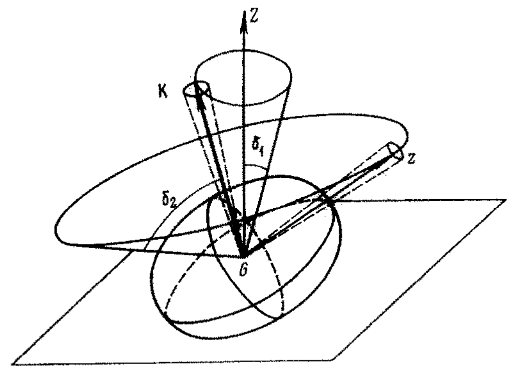

Fig.6

Presence of the three cyclic coordinates enables us, using the Routh algorithm, to reduce the problem of ellipsoid motion to the analysis of a system with two degrees of freedom. For convenience of the analysis of motions we express the Hamiltonian of the reduced system in terms of Andoyer's canonical variables /19,20/. In Fig.5 $GXYZ$ is a system of coordinates with origin at the ellipsoid center of mass and axes parallel to the corresponding axes of the fixed coordinate system $OXYZ$, $\mathbf{K}$ is the vector of the ellipsoid moment of momentum relative to its center of mass, and GMN is a plane normal to $\mathbf{K}$ which intersects the planes GXY and Gxy along the straight lines GM and GN, respectively. The canonical variables are L, I_2, I_3, l, φ_2, φ_3. The meaning of angular variables l, φ_2, φ_3 is clear from Fig.5, and their corresponding momenta are obviously $L = K\cos\delta_2$, $I_2 = K$, $I_3 = K\cos\delta_1$.

The momentum I_2 is the length of the moment of momentum vector, and L and I_3 are its projections, respectively, on the ellipsoid axis Gz and the vertical.

In the case of an ellipsoid only slightly different from a sphere the Hamiltonian of the reduced system in Andoyer's variables is of the form

$$H(L, L_2, l, \varphi_2) = \frac{I_2^2}{2A} + \frac{1}{2A}\left[\frac{c-a}{R}L^2 + \frac{b-a}{R}(I_2^2 - L^2)\cos^2 l\right] + g[(b-a)a_{32}^2 + (c-a)a_{33}^2] + \ldots \quad (10.1)$$

$$a_{32} = \cos\delta_1 \sin\delta_2 \cos l + \sin\delta_1 \cos\delta_2 \cos l \cos\varphi_2 - \sin\delta_1 \sin l \sin\varphi_2$$

$$a_{33} = \cos\delta_1 \cos\delta_2 - \sin\delta_1 \sin\delta_2 \cos\varphi_2$$

$$\cos\delta_1 = I_3/I_2, \qquad \cos\delta_2 = L/I_2$$

When $\varepsilon = 0$, motion of the ellipsoid relative to its center of mass is defined by the Hamiltonian $H_0 = I_2^2/2A$ and represents the uniform rotation at the velocity $\varphi_2^{\cdot} = I_2/A$ about the moment of momentum vector of fixed magnitude and direction. We assume this to be the unperturbed motion and shall investigate the perturbed motion (with $\varepsilon \neq 0$)) of the ellipsoid by the method of averaging /21/.

In the system of canonical differential equations with Hamiltonian (10.1) the variables L, I_2, l are slow and φ_2 is rapid. The first approximation solution is obtained by the averaging of Hamiltonian (10.1) with respect to variable φ_2 rejecting terms of the second and higher orders of smallness with respect to ε. We find that in the first approximation the motion is defined by a system of equations with the Hamiltonian

$$\Gamma = \frac{I_2^2}{2A} + \frac{b+c-2a}{2} g \sin^2\delta_1 + \frac{\varkappa}{2a}\left[\frac{c-a}{R}L^2 + \frac{b-a}{R}(I_2^2 - L^2)\cos^2 l\right], \quad (10.2)$$

$$\varkappa = 1 + \frac{5gA^2}{2I_2^2 R}(2 - 3\sin^2\delta_1)$$

In the first approximation the variables L, I_2, l vary with time as in the Euler–Poinsot motion, if in the latter the quantity $\tau = \varkappa t$ is taken as the time. The moment of momentum vector is in the first approximation of constant magnitude and slowly precesses about the vertical at constant angular velocity $\partial\Gamma/\partial I_3$, remaining at the constant angle $\delta_1 = \arccos(I_3/I_2)$ to it. Projection of the moment of momentum vector on the Gz axis of the ellipsoid slowly varies with time at the rate $L^{\cdot} = -\partial\Gamma/\partial l$, which results in a slow variation of angle δ_2 between the Gz axis of the ellipsoid and the moment of momentum vector. The ellipsoid rapidly rotates about the moment of momentum vector with a slow varying angular velocity $\varphi_2^{\cdot} = \partial\Gamma/\partial I_2$; it also rotates about its Gz axis at the slow varying in time angular velocity $l^{\cdot} = \partial\Gamma/\partial L$. In a time interval of order ε^{-1} the slow variables L, I_2, l are determined in the first approximation with an error of order ε, while the error of the rapid variable φ_2 determination is of order unity. For a particular selection of initial data for which $\varkappa$ vanishes, the quantities L and l are, besides I_2, also constant in the first approximation. In that case the instant vector of angular velocity is constant relative to the moving ellipsoid (and to the absolute space).

More meaningful results can be obtained by taking as unperturbed instead of the motion of the sphere, the more complex motion defined by the Hamiltonian (10.1) averaged with respect to the rapid and slow variables φ_2 and l.

Let us represent Hamiltonian (10.1) in the form

$$H = H_0(I_2) + H_1(L, I_2) + H_2(L, I_2, l, \varphi_2) + \ldots \quad (10.3)$$

$$H_0 + H_1 = \frac{I_2^2}{2A} + \frac{b+c-2a}{2} g \sin^2\delta_1 + \frac{\varkappa}{4A}\left[2\frac{c-a}{R}L^2 + \frac{b-a}{R}(I_2^2 - L^2)\right] \quad (10.4)$$

Function H_2 in (10.3) is of the first order of smallness with respect to ε, and its value averaged with respect to φ_2 and l is zero. The dots denote terms of order of smallness higher than the first.

As the unperturbed motion of the ellipsoid we take the motion defined by the Hamiltonian $H_0 + H_1$. In the unperturbed motion

$$L^{\cdot} = 0, \; I_2^{\cdot} = 0 \quad (10.5)$$

$$l^{\cdot} = \partial H_1/\partial L = \omega_1, \; \varphi_2^{\cdot} = I_2/A + \partial H_1/\partial I_2 = \omega_2$$

In the case of unperturbed motion L and I_2 are constant, and taking into account also the invariability of I_3, we find that in unperturbed motion angles δ_1 and δ_2 are also constant. Thus in unperturbed motion the constant length vector $\mathbf{K}$ of the ellipsoid moment of momentum slowly precesses at constant angular velocity $\omega_3 = \partial H_1/\partial I_3$ about the vertical, while remaining at the constant angle $\delta_1 = \arccos(I_3/I_2)$. The ellipsoid itself effects (when $\varkappa \neq 0$) a regular precession about vector $\mathbf{K}$ and rotates at constant angular velocity ω_1 about its Gz axis which remains at constant angle $\delta_2 = \arccos(L/I_2)$ to the moment of momentum vector, and rotating about it at constant angular velocity ω_2.

Let us show, as in /22/, the stability of the described above unperturbed motion of the ellipsoid with respect to variables L, I_2 in the case of small perturbations of Hamiltonian (10.4). For this it is necessary to ascertain the nondegeneracy of the Hessian of function (10.4) /22/. We have

$$\begin{vmatrix} \frac{\partial^2 (H_0+H_1)}{\partial L^2} & \frac{\partial^2 (H_0+H_1)}{\partial L\,\partial I_2} \\ \frac{\partial^2 (H_0+H_1)}{\partial I_2\,\partial L} & \frac{\partial^2 (H_0+H_1)}{\partial I_2^2} \end{vmatrix} = \frac{\varkappa}{2A^2R}(2c-a-b)+\ldots \quad (10.6)$$

Since the quantity $\varkappa$ is assumed nonzero for the unperturbed motion, the determinant (10.6) is nonzero for fairly small ε, provided the ellipsoid semiaxes satisfy the inequality

$$a+b\neq 2c \quad (10.7)$$

It has, thus, been shown that in conformity with /22/ in the case of an ellipsoid only slightly differing from a sphere that, when condition (10.7) is satisfied, the variation of L and I_2 over an infinite time interval is arbitrarily small.

A more exact meaning of the above statement is as follows. When inequality (10.7) is satisfied, then for any $\mu>0$ there exists a $\varepsilon^*>0$ such that for all ε from the interval $0<\varepsilon<\varepsilon^*$ for perturbed motion $|L(t)-L(0)|<\mu$, $|I_2(t)-I_2(0)|<\mu$ at any t.

This means that the length K of the moment of momentum vector and its angle δ_1 to the vertical remain close to their initial values at all t. The angle δ_2 between the moment of momentum vector and the ellipsoid axis Gz (Fig.6) always remains close to its initial value.

The frequencies ω_i $(i=1,2,3)$ will always remain close to their initial values, but the angles $l(t), \varphi_2(t), \varphi_3(t)$ that correspond to them are generally not close to their values in unperturbed motion calculated for one and the same instant of time.

It should be stressed that the analysis in Sect.10 was carried out on the assumption that the ellipsoid is at all times in contact with the plane on which it moves. This assumption is valid in the case of an ellipsoid only slightly differing from a sphere moving at some specified initial angular velocity, since it follows from the theorem on the motion of the center of mass, and because the quantity $\ddot{z}_G$ is arbitrarily small for fairly small ε that the normal reaction of the plane is always positive (and close to the ellipsoid weight). The latter means that the ellipsoid does not jump over the plane at any time, but is in contact with it at all times.

REFERENCES

1. ROUTH E.J., The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies. London, Macmillan, 1884.
2. CHAPLYGIN S.A., Investigations on Dynamics of Nonholonomic Systems. Moscow — Leningrad, Gostekhizdat, 1949.
3. VORONETS P.V., The Equations of Motion of a Solid Body Rolling without Slip on a Fixed Plane. Kiev, Printing House of the St. Vladimir Univ., 1903.
4. APPEL P., Traité de Mécanique Rationelle, Vol.2. Paris, Gauthier-Villars, 1953 (Russian translation, Moscow, Fizmatgiz, 1960).
5. GRAY A., A Treatise on Gyrostatics and Rotational Motion. London, Macmillan, 1918.
6. LEVI-CIVITA T. and AMALDI U., Lezioni di Meccanica Razuinale, Vol.2, No.2, Bologna, Zanichelli, 1927 (Russian translation, Moscow, Izd. Inostr. Lit., 1951).
7. MUSHTARI Kh.M., ON the rolling of a heavy solid body of revolution on a fixed horizontal plane. Matem. Sb., Vol.39, No.1—2, 1932.
8. NEIMARK Iu.M. and FUFAEV N.A., Dynamics of Nonholonomic Systems. Moscow, NAUKA, 1967.
9. RUMIANTSEV V.V. and KARAPETIAN A.V., Stability of motion of nonholonomic systems. In: Itogi Nauki i Tekhniki: Obshchaia Mekhanika, Vol.3, Moscow, VINITI, 1976.
10. KARAPETIAN A.V., On permanent rotations of a heavy solid body on a horizontal plane and their stability. In: Annotations of Transac. of the 5-th Congress on Theor. and Appl. Mechanics. Alma-Ata, NAUKA, 1981.
11. RUMIANTSEV V.V., On stability of rotation of a heavy gyrostat on a horizontal plane. Izv. Akad. Nauk SSSR, MTT, No.4, 1980.
12. KARAPETIAN A.V., On the problem of steady motion stability of nonholonomic systems. PMM, Vol.44, No.3, 1980.
13. KARAPETIAN A.V., On stability of steady motions of a heavy solid body on an absolutely smooth horizontal plane. PMM, Vol.45, No.3, 1981.
14. KARAPETIAN A.V., On permanent rotations of a heavy solid body on an absolutely rough horizontal plane. PMM, Vol.45, No.5, 1981.
15. PARS L.A., A Treatise on Analytical Dynamics. London, Heineman, 1965. (Russian translation, Mekhanika, Sb. Perev. Inostr. Statei. Moscow, NAUKA, 1971).

16. CESARI L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations. 3rd ed. Berlin, N.Y., Springer-Verlag, 1959 and 1971. (Russian translation: Moscow, MIR, 1964).
17. RASHEVSKII P.K., A Course of Differential Geometry, Moscow, Gostekhizdat, 1956.
18. ANDOYER M.H., Course de Méchanique Céleste. Paris, Gauthier-Villars, Vol.1, 1923, and Vol. 2, 1926.
19. ARKHANGEL'SKII Iu.A., Analytical Dynamics of the Solid Body. Moscow, NAUKA, 1977.
20. VOLOSOV V.M., Averaging in systems of ordinary differential equations. Uspekhi Matem. Nauk. Vol.17, No.6, 1962.
21. BARRAR R., A proof of convergence of the Poincaré—von Zeipel procedure in celestial mechanics. Amer. J. Math., Vol.88, No., 1966.

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